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Analytically solvable regions for coupling parameters of coupled nonlinear equations

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Abstract. A method is presented by which analytically solvable N-coupled nonlinear Schrödinger and coupled quadratic equations are found for which the nonlinear coupling parameters of the equations can take up wide ranges of values and cover many regions. Specific results for $N = 2$ and 3 are presented. These results have potentially useful applications to problems in nonlinear optics and the dynamics of multispecies Bose–Einstein condensates.

1. Introduction

The problem of integrability and non-integrability of nonlinearly coupled dynamical systems with $N(>1)$ degrees of freedom has been a subject of considerable interest for many years [1]. One of the two coupled dynamical systems we shall discuss in this paper is closely related to the coupled nonlinear Schrödinger (CNLS) equations which have applications in many physical problems, especially in nonlinear optics [2]. The other is an analogue of the CNLS system except that the nonlinear coupling is quadratic instead of cubic, and we call it coupled quadratic (CQ) equations. CQ equations may not have any direct physical applications at the present time, but they serve to illustrate the generality and possible extension of the method for finding analytic solutions for coupled nonlinear equations which we give in this paper. We present analytic coupled solitary-wave solutions for the CNLS and CQ equations that show that there are many *regions* where the coupling parameters can change continuously over wide ranges of values for which the coupled equations are analytically solvable. For these coupling parameters, the equations may not be completely integrable, but are analytically solvable for the specific initial conditions prescribed by their analytic solutions. As is well known, only very few special values of coupling parameters satisfy the integrability requirements. One of the objectives of this paper is to focus attention away from integrable to analytically solvable cases because analytic solutions for the latter can be applied to many useful cases where the coupling parameters can assume wide ranges of values. Although we are restricted to using the initial conditions prescribed by these analytic solutions, they can often be achieved experimentally without too much difficulty. We begin by introducing two CNLS equations that arise in nonlinear optics.

When two optical waves of different frequencies copropagate in a medium and interact nonlinearly through the medium, the propagation equation for the slowly varying complex amplitude $\phi_m(z,t)$ of the mth electric field can be written as [2]

$$\mathrm{i}\phi_{mz} + \mathrm{i}\beta_{1m}\phi_{mt} - \frac{\beta_{2m}}{2}\phi_{mtt} + \frac{\mathrm{i}\alpha_m}{2}\phi_m + \gamma_m(f_{mm}|\phi_m|^2 + 2f_{mm'}|\phi_{m'}|^2)\phi_m = 0 \qquad m = 1,2 \quad m' \neq m \tag{1}$$

where $\beta_{1m} = 1/v_{gm}$, v_{gm} is the group velocity, β_{2m} is the group-velocity dispersion (GVD) coefficient, α_m is the loss coefficient, γ_m is the nonlinear coefficient and $f_{mm'}$ is the overlap integral and where the subscripts in z and t denote derivatives with respect to z and t as opposed to the subscript m for different components. The medium is said to exhibit normal dispersion if $\beta_2 > 0$, anomalous dispersion if $\beta_2 < 0$.

If the nonlinear coupling is between two polarization components $\phi_m(z, t)$, $m = 1, 2$, of a wave at some central frequency, the propagation equations are [2]

$$\mathrm{i}\phi_{1z} + \mathrm{i}\beta_{11}\phi_{1t} - \frac{\beta_2}{2}\phi_{1tt} + \frac{\mathrm{i}\alpha}{2}\phi_1 + \gamma[(|\phi_1|^2 + p|\phi_2|^2)\phi_1 + q\phi_1^*\phi_2^2\,\mathrm{e}^{-2\mathrm{i}\Delta\beta z}] = 0$$
$$\mathrm{i}\phi_{2z} + \mathrm{i}\beta_{12}\phi_{2t} - \frac{\beta_2}{2}\phi_{2tt} + \frac{\mathrm{i}\alpha}{2}\phi_2 + \gamma[(|\phi_2|^2 + p|\phi_1|^2)\phi_2 + q\phi_2^*\phi_1^2\,\mathrm{e}^{2\mathrm{i}\Delta\beta z}] = 0 \qquad (2)$$

where $\Delta\beta = \beta_{11} - \beta_{12}$ is the wavevector mismatch due to, for example, the birefringence of the medium through which the wave propagates and the parameters p and q satisfy $p + q = 1$.

If the two coupled waves or components propagate with approximately the same group velocity v, then $\mathrm{i}\beta_{1m}\phi_{mt}$ terms in equations (1) and (2) can be eliminated by the transformation $t \to t - z/v$ and equations (1) and (2) can be regarded together by considering the following N-coupled nonlinear Schrödinger-like equations for the case $N = 2$

$$\mathrm{i}\phi_{mz} + \phi_{mtt} + \kappa_m\phi_m + \left(\sum_{n=1}^{N} p_{mn}|\phi_n|^2\right)\phi_m + \left(\sum_{n=1}^{N} q_{mn}\phi_n^2\right)\phi_m^* = 0 \qquad m = 1, \ldots, N \qquad (3)$$

where p, q and κ are parameters characteristic of the medium or the following coupled equations

$$\mathrm{i}\psi_{mz} + \psi_{mtt} + \left(\sum_{n=1}^{N} p_{mn}|\psi_n|^2\right)\psi_m + \left(\sum_{n=1}^{N} q_{mn}\psi_n^2\,\mathrm{e}^{2\mathrm{i}\kappa_n z}\right)\psi_m^*\,\mathrm{e}^{-2\mathrm{i}\kappa_m z} = 0 \qquad m = 1, \ldots, N \qquad (4)$$

which can be transformed into (3) with substitutions $\psi_m = \phi_m \exp(-\mathrm{i}\kappa_m z)$.

We first search for the stationary-wave solution of the form

$$\phi_m(z, t) = x_m(t)\exp(\mathrm{i}\Omega z) \qquad (5)$$

where Ω is a real constant and $x_m(t)$ are real functions of t only. Equation (3) reduces to the following, which may be called associated dynamical CNLS equations

$$\ddot{x}_m - A_m x_m + \left(\sum_{n=1}^{N} b_{mn}x_n^2\right)x_m = 0 \qquad m = 1, \ldots, N \qquad (6)$$

where $\dot{x}$ denotes $\mathrm{d}x/\mathrm{d}t$ and where

$$A_m = \Omega - \kappa_m \qquad b_{mn} = p_{mn} + q_{mn}. \qquad (7)$$

To eliminate the permutation symmetry, we arrange equation (6) such that $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_N$. Since equations (3) and (4) are invariant under a Galilean transformation, travelling waves can be constructed from (5) by replacing $\phi_m(z, t)$ by

$$\phi_m(z, t - z/v)\exp\{\mathrm{i}[t - z/(2v)]/(2v)\} \qquad (8)$$

where v is the velocity of waves.

Comparing equation (3) with the complex conjugates of equation (1) shows that we can identify negative (positive) values of b_{jk}, $k = 1, \ldots, N$, in equation (6) with the normal (anomalous) GVD region for ϕ_j. The special integrable case $N = 2$, $A_j = 0$ and $b_{jk} = 1$ for $j, k = 1, 2$ is associated with the known integrable case of equation (1) first given by Manakov

[3]. Various solitary-wave solutions for this case that consist of the so-called bright and dark solitary waves, periodic (elliptic) waves and waves of other forms, have been presented [4–14]. Other values of b for which coupled equations are integrable have been given in [15]. The coupled equations of (6) have been of interest and studied in nonlinear dynamics for many years and are known to be integrable for a number of specific values of A and b [1].

Let us refer to the space spanned by N^2 real values of the nonlinear coupling parameters $b_{jk},\ j, k = 1, \ldots, N$, as the b-space. Instead of asking whether for some particular point of this b-space equation (6) is integrable, the key idea behind the results presented in this paper is to ask whether it is possible to postulate N analytic solutions for $x_1, \ldots, x_N$, with variable parameters and to find points or regions in the b-space for these solutions to hold, so that for these points or regions, equation (6) is analytically solvable. In this paper, we show that there are many *regions* in the b-space where the values for b can change continuously over wide ranges and for which the coupled equations are analytically solvable. We present a method and prescription for obtaining such regions and present specifically 16 analytically solvable regions and the corresponding coupled solitary waves for the case $N = 2$ for the CNLS equations. The solitary waves are given in terms of Jacobian elliptic functions and are thus generally of the periodic type. The aperiodic type that corresponds to the special case $k^2 = 1$, where k is the modulus of the elliptic functions, may be of greater interest. However, it will be clear from our results that restricting the use of waves to those of the aperiodic type reduces access to only a small part of the analytically solvable b-space.

The analytically solvable regions for $N > 2$ are also of interest and can be obtained from our prescription. Possible advantages in increasing the number of interacting fields from, say, two to three, will be pointed out. For these cases, we shall present results for aperiodic waves for the CQ as well as CNLS equations.

We present our method and prescription in section 2. In section 3, we present sixteen analytically solvable regions for CNLS equations for $N = 2$ with periodic-wave solutions. In section 4, we discuss the special case of aperiodic waves and present a more compact formulation for the solutions and for finding the analytically solvable regions for N-CNLS and CQ equations. A summary is given in section 5.

2. A prescription for finding the analytically solvable regions for CNLS equations

Considering equation (6), we make the ansatz that $x_1(t), \ldots, x_N(t)$ is expressible in terms of N of the $(2n+1)$ Lamé functions of order n [16], with repetition allowed (i.e. the same function for different x) for $n = 1, \ldots, N-1$ and without repetition for $n = N$.

Let $h_j^{(n)},\ j = 1, \ldots, 2n+1$, arranged in descending order of magnitude, denote the characteristic values and $f_j^{(n)}(u)$ the corresponding characteristic function (Lamé function), of Lamé equations of order n, $\mathrm{d}^2 y/\mathrm{d}u^2 + \{h - n(n+1)k^2 \mathrm{sn}^2(u, k)\}y = 0$. We make the ansatz that

$$x_1(t) = \sqrt{C_1} f_p^{(n)}(\alpha t) \qquad x_2(t) = \sqrt{C_2} f_q^{(n)}(\alpha t), \ldots, x_N(t) = \sqrt{C_N} f_s^{(n)}(\alpha t) \tag{9}$$

is a solution of equation (6), where $n = 1, \ldots, N,\ p, q, \ldots, s = 1, \ldots, 2n+1,\ p \leqslant q \leqslant \cdots \leqslant s$ for $n = 1, \ldots, N-1$ and $p < q < \cdots < s$ for $n = N$. Since $x_1(t), x_2(t), \ldots, x_N(t)$ are assumed real, we require that $C_1, C_2, \ldots, C_N$, are real and positive. Substitutions of the ansatz (9) into equation (6) result in algebraic equations for b, A, C, α and k^2 which can be expressed in a compact way in terms of three matrices $\boldsymbol{\Gamma}$, $\mathbf{B}$ and $\mathbf{D}$ which we define in the following. First, we express the square of the jth Lamé function of order n in a power series in $s \equiv \mathrm{sn}(u, k)$ as

$$[f_j^{(n)}(u)]^2 = \sum_{i=1}^{n+1} a_{ij}^{(n)} s^{2(i-1)} \qquad j = 1, \ldots, 2n+1. \tag{10}$$

We form an $(n+1)\times(2n+1)$ matrix $\mathbf{a} = [a_{ij}^{(n)}]$. Define $\Gamma = [c_{ij}]$ to be an $((n+1)\times N)$ matrix where $c_{i1} = a_{ip}^{(n)}C_1$, $c_{i2} = a_{iq}^{(n)}C_2, \ldots, c_{iN} = a_{is}^{(n)}C_N$, $i = 1, \ldots, n+1$ and where C_j are the amplitudes in (9). $\mathbf{B} = [b_{ij}]$, $i, j = 1, \ldots, N$ is an $(N\times N)$ matrix, where b_{ij} are the nonlinear coupling parameters in equation (6). $\mathbf{D} = [d_{ij}^{(n)}]$, $i = 1, \ldots, n+1$, $j = 1, \ldots, N$ is an $((n+1)\times N)$ matrix, where $d_{11}^{(n)} = A_1 + h_p^{(n)}\alpha^2$, $d_{12}^{(n)} = A_2 + h_q^{(n)}\alpha^2, \ldots, d_{1N}^{(n)} = A_N + h_s^{(n)}\alpha^2$, $d_{2j}^{(n)} = -n(n+1)k^2\alpha^2$, $d_{3j}^{(n)} = \cdots = d_{n+1,j}^{(n)} = 0$, $j = 1, \ldots, N$ and where A_j are the parameters in equation (6), $h_j^{(n)}$ the characteristic values of the Lamé equation and α the scaling parameter in (9). The algebraic equations that need to be satisfied for (9) to be a solution of equation (6) can now be expressed conveniently as

$$\Gamma\mathbf{B}^{\mathrm{T}} = \mathbf{D} \tag{11}$$

where $\mathbf{B}^{\mathrm{T}}$ denotes the transposed matrix of $\mathbf{B}$. For $N = 2$, we can readily solve equation (11) for $n = 1$, $p, q = 1, 2, 3$, $p \leqslant q$ and for $n = 2$, $p, q = 1, \ldots, 5$ and $p < q$, and obtain 16 analytically solvable regions in b-space or 16 sets of explicit expressions of b in terms of the arbitrary amplitudes C_1 and C_2 of the waves and in terms of A_1, A_2, k^2 and α^2. The modulus k of the elliptic functions that express the Lamé functions, which is in the range $0 < k^2 \leqslant 1$ unless otherwise specified, may be considered as another variable parameter. For some of these analytically solvable regions, the values of A in equation (6) are constrained, but for others, they are free to take up any values unless they are physically constrained.

3. Analytically solvable regions for two CNLS equations

Treating the amplitudes C_1 and C_2 for x_1 and x_2, the modulus k, the scaling parameter α and in some cases A_1 and A_2, as variable parameters, the sixteen analytically solvable regions in b-space for equation (6), $N = 2$, are given in (I)–(XVI) later together with the analytic solutions for x_1 and x_2. Using transformation (8), these are thee regions of b for which equation (3) or (4) have analytic coupled solitary-wave solutions. We denote $G_\pm = 1 + k^2 \pm (1 - k^2 + k^4)^{1/2}$.

(I) $x_1 = \sqrt{C_1}\,\mathrm{sn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\,\mathrm{sn}(\alpha t, k)$

$A_1 = A_2 = -(1 + k^2)\alpha^2$

$b_{11}/b_{21} = b_{12}/b_{22} = 1 \qquad b_{11}C_1 + b_{12}C_2 = -2k^2\alpha^2.$

(II) $x_1 = \sqrt{C_1}\,\mathrm{cn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\,\mathrm{cn}(\alpha t, k)$

$A_1 = A_2 = (2k^2 - 1)\alpha^2.$

For $b_{11} > b_{21} \qquad b_{22} > b_{12}$

$C_1 = 2k^2\alpha^2(b_{22} - b_{12})\Delta^{-1} \qquad C_2 = 2k^2\alpha^2(b_{11} - b_{21})\Delta^{-1}$

where $\Delta = b_{11}b_{22} - b_{12}b_{21}$.

For $b_{11}/b_{21} = b_{12}/b_{22} = 1 \qquad b_{11}C_1 + b_{12}C_2 = 2k^2\alpha^2.$

(III) $x_1 = \sqrt{C_1}\,\mathrm{dn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\,\mathrm{dn}(\alpha t, k)$

$A_1 = A_2 = (2 - k^2)\alpha^2.$

For $b_{11} > b_{21} \qquad b_{22} > b_{12}$

$C_1 = 2\alpha^2(b_{22} - b_{12})\Delta^{-1} \qquad C_2 = 2\alpha^2(b_{11} - b_{21})\Delta^{-1}$

where $\Delta = b_{11}b_{22} - b_{12}b_{21}$.

For $b_{11}/b_{21} = b_{12}/b_{22} = 1 \qquad b_{11}C_1 + b_{12}C_2 = 2\alpha^2.$

(IV) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\mathrm{cn}(\alpha t, k)$

$b_{11} = [A_1 + (1-k^2)\alpha^2]C_1^{-1} \qquad b_{12} = [A_1 + (1+k^2)\alpha^2]C_2^{-1}$

$b_{21} = [A_2 + (1-2k^2)\alpha^2]C_1^{-1} \qquad b_{22} = [A_2 + \alpha^2]C_2^{-1}.$

(V) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\mathrm{dn}(\alpha t, k)$

$b_{11} = k^2[A_1 - (1-k^2)\alpha^2]C_1^{-1} \qquad b_{12} = [A_1 + (1+k^2)\alpha^2]C_2^{-1}$

$b_{21} = k^2[A_2 - (2-k^2)\alpha^2]C_1^{-1} \qquad b_{22} = [A_2 + k^2\alpha^2]C_2^{-1}.$

(VI) $x_1 = \sqrt{C_1}\mathrm{cn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\mathrm{dn}(\alpha t, k)$

$b_{11} = -k^2k'^{-2}[A_1 - \alpha^2]C_1^{-1} \qquad b_{12} = k'^{-2}[A_1 + (1-2k^2)\alpha^2]C_2^{-1}$

$b_{21} = -k^2k'^{-2}[A_2 + (2-k^2)\alpha^2]C_1^{-1}$

$b_{22} = k'^{-2}[A_2 - k^2\alpha^2]C_2^{-1} \qquad 0 < k^2 < 1.$

(VII) $x_1 = \sqrt{C_1}[\frac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t, k)] \qquad x_2 = \sqrt{C_2}\mathrm{sn}(\alpha t, k)\mathrm{cn}(\alpha t, k)$

$b_{11} = 9G_-^{-2}[A_1 + 2G_+\alpha^2]C_1^{-1} \qquad b_{12} = 6k^2G_-^{-1}[A_1 + (2G_+ - G_-)\alpha^2]C_2^{-1}$

$b_{21} = 9G_-^{-2}[A_2 + (4+k^2)\alpha^2]C_1^{-1} \qquad b_{22} = 6k^2G_-^{-1}[A_2 + (4+k^2-G_-)\alpha^2]C_2^{-1}$

$A_1 = 2\alpha^2(2G_+G_- - 3k^2G_+ - 1)/(3k^2 - 2G_-)$

$A_2 = \alpha^2[2(4+k^2)G_- - 3k^2(4+k^2) - 2]/(3k^2 - 2G_-).$

(VIII) $x_1 = \sqrt{C_1}[\frac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t, k)] \qquad x_2 = \sqrt{C_2}\mathrm{sn}(\alpha t, k)\mathrm{dn}(\alpha t, k)$

$b_{11} = 9G_-^{-2}[A_1 + 2G_+\alpha^2]C_1^{-1} \qquad b_{12} = 6k^2G_-^{-1}[A_1 + (2G_+ - G_-)\alpha^2]C_2^{-1}$

$b_{21} = 9G_-^{-2}[A_2 + (1+4k^2)\alpha^2]C_1^{-1} \qquad b_{22} = 6k^2G_-^{-1}[A_2 + (1+4k^2-G_-)\alpha^2]C_2^{-1}$

$A_1 = 2\alpha^2[G_-(2G_+ - G_-) - 3G_+]/(3 - 2G_-)$

$A_2 = \alpha^2[2G_-(1+4k^2-G_-) - 3(1+4k^2)]/(3 - 2G_-).$

(IX) $x_1 = \sqrt{C_1}[\frac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t, k)] \qquad x_2 = \sqrt{C_2}\mathrm{cn}(\alpha t, k)\,\mathrm{dn}(\alpha t, k)$

$b_{11} = -9G_-^{-1}\Delta^{-1}\{(1+k^2)A_1 + 2[G_+(1+k^2) - 3k^2]\alpha^2\}C_1^{-1}$

$b_{12} = 6k^2\Delta^{-1}\{A_1 + (2G_+ - G_-)\alpha^2\}C_2^{-1}$

$b_{21} = -9G_-^{-1}\Delta^{-1}\{(1+k^2)A_2 + (1-4k^2+k^4)\alpha^2\}C_1^{-1}$

$b_{22} = 6k^2\Delta^{-1}\{A_2 + (1+k^2-G_-)\alpha^2\}C_2^{-1}$

where $\Delta = 6k^2 - G_-(1+k^2)$

$A_1 = 2\alpha^2\{G_-(2G_+ - G_-) - 3[G_+(1+k^2) - 3k^2]\}/[3(1+k^2) - 2G_-]$

$A_2 = \alpha^2\{2G_-(1+k^2-G_-) - 3(1-4k^2+k^4)\}/[3(1+k^2) - 2G_-].$

(X) $x_1 = \sqrt{C_1}[\frac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t, k)] \qquad x_2 = \sqrt{C_2}[\frac{1}{3}G_+ - k^2\mathrm{sn}^2(\alpha t, k)]$

$b_{11} = -9G_-^{-1}\Delta^{-1}(A_1 + G_+\alpha^2\}C_1^{-1} \qquad b_{12} = 9G_+^{-1}\Delta^{-1}\{A_1 + (2G_+ - G_-)\alpha^2\}C_2^{-1}$

$b_{21} = -9G_-^{-1}\Delta^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1} \qquad b_{22} = 9G_+^{-1}\Delta^{-1}(A_2 + G_-\alpha^2\}C_2^{-1}$

where $\Delta = G_+ - G_-$

$A_1 = \alpha^2(2G_+G_- - G_+^2 - G_-^2)\Delta^{-1} \qquad A_2 = \alpha^2(-2G_+G_- + G_+^2 + G_-^2)\Delta^{-1}.$

(XI) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k)\mathrm{cn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\mathrm{sn}(\alpha t, k)\mathrm{dn}(\alpha t, k)$

$b_{11} = b_{21} = 6\alpha^2k^4k'^{-2}C_1^{-1} \qquad b_{12} = b_{22} = -6\alpha^2k^2k'^{-2}C_2^{-1}$

$A_1 = -(4+k^2)\alpha^2 \qquad A_2 = -(1+4k^2)\alpha^2 \qquad 0 < k^2 < 1.$

(XII) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k)\mathrm{cn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\mathrm{cn}(\alpha t, k)\mathrm{dn}(\alpha t, k)$

$$b_{11} = \{(1+k^2)A_1 + (4-k^2+k^4)\alpha^2\}C_1^{-1} \qquad b_{12} = \{A_1 + (4+k^2)\alpha^2\}C_2^{-1}$$
$$b_{21} = \{(1+k^2)A_2 + (1-4k^2+k^4)\alpha^2\}C_1^{-1} \qquad b_{22} = \{A_2 + (1+k^2)\alpha^2\}C_2^{-1}$$
$$A_1 = (5k^2-4)\alpha^2 \qquad A_2 = (5k^2-1)\alpha^2.$$

(XIII) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k)\mathrm{cn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}[\frac{1}{3}G_+ - k^2\mathrm{sn}^2(\alpha t, k)]$

$$b_{11} = 6k^2G_+^{-1}\{A_1 + (4+k^2-G_+)\alpha^2\}C_1^{-1} \qquad b_{12} = 9G_+^{-2}\{A_1 + (4+k^2)\alpha^2\}C_2^{-1}$$
$$b_{21} = 6k^2G_+^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1} \qquad b_{22} = 9G_+^{-2}\{A_2 + 2G_-\alpha^2\}C_2^{-1}$$
$$A_1 = \alpha^2\{3k^2(4+k^2) - 2G_+(4+k^2-G_+)\}/(2G_+ - 3k^2)$$
$$A_2 = \alpha^2\{6k^2G_- - 2G_+(2G_- - G_+)\}/(2G_+ - 3k^2).$$

(XIV) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k)\mathrm{dn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}\mathrm{cn}(\alpha t, k)\mathrm{dn}(\alpha t, k)$

$$b_{11} = \{(1+k^2)A_1 + (1-k^2+4k^4)\alpha^2\}C_1^{-1} \qquad b_{12} = \{A_1 + (1+4k^2)\alpha^2\}C_2^{-1}$$
$$b_{21} = \{(1+k^2)A_2 + (1-4k^2+k^4)\alpha^2\}C_1^{-1} \qquad b_{22} = \{A_2 + (1+k^2)\alpha^2\}C_2^{-1}$$
$$A_1 = (5-4k^2)\alpha^2 \qquad A_2 = (5-k^2)\alpha^2.$$

(XV) $x_1 = \sqrt{C_1}\mathrm{sn}(\alpha t, k)\mathrm{dn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}[\frac{1}{3}G_+ - k^2\mathrm{sn}^2(\alpha t, k)]$

$$b_{11} = 6k^2G_+^{-1}\{A_1 + (1+4k^2-G_+)\alpha^2\}C_1^{-1} \qquad b_{12} = 9G_+^{-2}\{A_1 + (1+4k^2)\alpha^2\}C_2^{-1}$$
$$b_{21} = 6k^2G_+^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1} \qquad b_{22} = 9G_+^{-2}\{A_2 + 2G_-\alpha^2\}C_2^{-1}$$
$$A_1 = \alpha^2\{3(1+4k^2) - 2G_+(1+4k^2-G_+)\}/(2G_+ - 3)$$
$$A_2 = \alpha^2\{6G_- - 2G_+(2G_- - G_+)\}/(2G_+ - 3).$$

(XVI) $x_1 = \sqrt{C_1}\mathrm{cn}(\alpha t, k)\mathrm{dn}(\alpha t, k) \qquad x_2 = \sqrt{C_2}[\frac{1}{3}G_+ - k^2\mathrm{sn}^2(\alpha t, k)]$

$$b_{11} = -6G_+k^2\Delta^{-1}\{A_1 + (1+k^2-G_+)\alpha^2\}C_1^{-1}$$
$$b_{12} = 9\Delta^{-1}\{(1+k^2)A_1 + (1-4k^2+k^4)\alpha^2\}C_2^{-1}$$
$$b_{21} = -6G_+k^2\Delta^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1}$$
$$b_{22} = 9\Delta^{-1}\{(1+k^2)A_2 + [2G_-(1+k^2) - 6k^2]\alpha^2\}C_2^{-1}$$

where $\Delta = (1+k^2)G_+^2 - 6G_+k^2$

$$A_1 = \alpha^2\{2G_+(1+k^2-G_+) - 3(1-4k^2+k^4)\}/[3(1+k^2) - 2G_+]$$
$$A_2 = \alpha^2\{2G_+(2G_- - G_+) - 6[G_-(1+k^2) - 3k^2)\}/[3(1+k^2) - 2G_+].$$

These results show many analytically solvable regions for the two coupled dynamical equation (6) and for two CNLS like equations (3) and (4). The explicit expressions (I)–(XVI) could open up more applications in optical communications. We note that some wavepairs can be in the 'mixed' GVD region; i.e. one wave in the normal while the other in the anomalous GVD region, some wavepairs can be in the normal or anomalous GVD regions for both waves; they are not always restricted for use in only those regions because depending on the choice of amplitudes and modulus, the same wavepair can be made to propagate as a solitary wavepair in optical media of different character. Prospects for experimental applications of these shape-preserving 'Jacobian elliptic wavetrains' have been greatly enhanced following a recent experimental observation [17] of the evolution of an arbitrarily shaped input optical pulse-train to the shape-preserving Jacobian elliptic pulse-train for Maxwell–Bloch equations.

If we restrict ourselves to using only aperiodic waves that correspond to $k^2 = 1$, then the analytically solvable regions are reduced in number and size considerably. The possible aperiodic solitary waves have the forms tanh $\alpha\xi$ and sech $\alpha\xi$, or the well known dark and bright solitary waves, for waves of order one and the forms $\operatorname{sech}^2 \alpha\xi - (2/3)$, $\tanh \alpha\xi \operatorname{sech} \alpha\xi$ and $\operatorname{sech}^2 \alpha\xi$, or the so-called red, white and blue solitary waves [18], for waves of order two.

We note that two of the three aperiodic solitary waves of order two were found previously by Tratnik and Sipe [8] and four of the five periodic solitary waves of order two by Kostov and Uzunov [13], for the case of CNLS equations for $N = 2$ and $b_{11} = b_{12} = b_{21} = b_{22}$. In these and previous cases, the problem had been approached with some specified values of the parameter b, for which analytic solutions might not exist or might exist only for some specific waves of very specific amplitudes. The role played by the entire sets of waves of order $n = 1, 2, \ldots$ for N CNLS equations was not made apparent. On the other hand, the generality for n, N and for b which we have been able to achieve has been realized because we have taken the approach from the point of view that is suggested by the algebraic equation (11) resulting from our ansatz (9) in which the values of b are left open. The subsequent results on the admissible coupled solitary waves have led to other interesting concepts which we mention later. First, let us further exemplify the usefulness of this approach in the following section.

4. Aperiodic waves and N coupled quadratic equations

We have seen how the method and prescription given by equations (9)–(11) can be used to find the analytically solvable regions for the nonlinear coupling parameters of N CNLS equations. Since the aperiodic waves are of particular interest and since for the case $k^2 = 1$, $(2n + 1)$ Lamé functions of order n reduce to $(n + 1)$ Lamé functions which are expressible in compact forms, we shall give a more compact formulation for finding analytically solvable regions for N CNLS equations for the special case that the N waves for $x_1, \ldots, x_N$ are not only aperiodic (i.e. $k^2 = 1$) but are also all different (we call complementary). Indeed, to get a better understanding of whether our method can be applied to other N coupled nonlinear equations, we introduce an analogous set of coupled equations which we call N *CQ* equations given by

$$\mathrm{i}\phi_{mz} + \phi_{mtt} + \kappa_m \phi_m + \left(\sum_{n=1}^{N} p_{mn}|\phi_n|\right)\phi_m + \left(\sum_{n=1}^{N} q_{mn}\phi_n\right)\phi_m^* = 0 \qquad m = 1, 2, \ldots, N. \tag{12}$$

The corresponding associated dynamical CQ equations are

$$\ddot{x}_m - A_m x_m + \left(\sum_{n=1}^{N} b_{mn} x_n\right) x_m = 0 \qquad m = 1, 2, \ldots, N \tag{13}$$

where we have used the same substitutions (5) and notations (7). We assume that equation (13) has been arranged such that $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_N$.

The $(2n + 1)$ Lamé functions of order n become, for $k^2 = 1$, $n + 1$ associated Legendre functions $P_n^m(x)$ of order n and degree $m = 0, 1, \ldots, n$, where $x = \tanh(u)$. We make the ansatz that

$$x_j = \sqrt{C_j}\, P_{N-1}^{j-1}[\tanh(\alpha t)] \qquad j = 1, 2, \ldots, N \tag{14}$$

where C_j is real and positive, for equation (6), but that

$$x_j = C_j P_{2(N-1)}^{2(j-1)}[\tanh(\alpha t)] \qquad j = 1, 2, \ldots, N \tag{15}$$

where C_j is real but not necessarily positive, for equation (13). Thus we express $x_1, \ldots, x_N$, in terms of N Lamé functions (for $k^2 = 1$) of order $N - 1$ for CNLS equations and in terms of a certain subset or N of the $(2N - 1)$ Lamé functions (for $k^2 = 1$) of order $2(N - 1)$, for CQ equations. As with equations (10) and (11), we define three matrices in the following but they all have the same dimension ($N \times N$). We first define $a_{ij}^{(N)}$ as follows. For equation (6), $a_{ij}^{(N)}$ is the coefficient of $x^{2(i-1)}$ in $[P_{N-1}^{j-1}(x)]^2$ when it is expanded as

$$[P_{N-1}^{j-1}(x)]^2 = \sum_{i=1}^{N} a_{ij}^{(N)} x^{2(i-1)} \tag{16}$$

while for equation (13), $a_{ij}^{(N)}$ is the coefficient of $x^{2(i-1)}$ in $P_{2(N-1)}^{2(j-1)}(x)$ when it is expressed as

$$P_{2(N-1)}^{2(j-1)}(x) = \sum_{i=1}^{N} a_{ij}^{(N)} x^{2(i-1)}. \tag{17}$$

Our first ($N \times N$) matrix is $\mathbf{\Gamma} = [c_{ij}]$ whose matrix elements $c_{ij} = a_{ij}^{(N)} C_j$, where C_j is the coefficient in (14) for equation (6) and is the coefficient in (15) for equation (13). Our second matrix is $\mathsf{B} = [b_{ij}]$, where b_{ij} are the nonlinear coupling parameters given in equation (6) or (13). Our third matrix is $\mathsf{D} = [d_{ij}]$, where $d_{1j} = A_j + [(N-1)N - (j-1)^2]\alpha^2]$, $d_{2j} = -(N-1)N\alpha^2$, $d_{3j} = d_{4j} = \cdots = d_{Nj} = 0$ for equation (6) and $d_{1j} = A_j + [(2N-2)(2N-1) - 4(j-1)^2]\alpha^2]$, $d_{2j} = -(2N-2)(2N-1)\alpha^2$, $d_{3j} = d_{4j} = \cdots = d_{N_j} = 0$ for equation (13).

Substitutions of the ansatz (14) or (15) into equations (6) or (13) lead to N^2 algebraic equations which can be expressed conveniently in terms of the three matrices, $\mathbf{\Gamma}$, B and D defined as

$$\mathbf{\Gamma}\mathsf{B}^{\mathrm{T}} = \mathsf{D}$$

or

$$\mathsf{B}^{\mathrm{T}} = \mathbf{\Gamma}^{-1}\mathsf{D} \tag{18}$$

where B^{T} denotes the transposed matrix of B. Provided that $\mathbf{\Gamma}^{-1}$ exists, equation (18) gives the set of parameters b_{ij} in equations (6) or (13) in terms of A_j given in those equations and in terms of the generally arbitrary amplitudes C_j in (14) or (15), i.e. equation (13) gives the N^2 nonlinear coupling parameters b_{ij} in terms of $2N$ variable parameters $A_j, C_j, j = 1, \ldots, N$ and the scaling parameter α and for these b_{ij}, equations (6) or (13) are analytically solvable.

If we are given a specific set of N^2 values of b_{ij} and asked whether $2N$ A_j and C_j (C_j must be >0 for equation (6)) and α can be found that yield solutions (14) and (15), the answer would be not generally unless the given values of b_{ij} fall into one of the analytically solvable regions, or in other words, unless $2N + 1$ values of A_j, C_j and α can be found such that they satisfy the N^2 equations, or, unless these given values of b_{ij} can be shown to be an integrable set in another way.

For the special case $b_{ij} = \varepsilon$, for all $i, j = 1, \ldots, N$, where $\varepsilon = +1$ or -1, the answer is affirmative and we can given the solution in a compact form. We write equation (18), in this case, as

$$\mathsf{a}\vec{C} = \vec{d} \tag{19}$$

where the ($N \times N$) matrix $\mathsf{a} = [a_{ij}^{(N)}]$, the N-dimensional column vector $\vec{C} = \mathrm{col}(C_1, C_2, \ldots, C_N)$ and the N-dimensional column vector $\vec{d} = \mathrm{col}(d_{11}, d_{21}, \ldots, d_{N1})$. The consistency requirement becomes $(N-1)$ equations on $A_2, \ldots, A_N$ which must be related to A_1 by

$$A_j = A_1 + (j-1)^2\alpha^2 \qquad j = 2, \ldots, N \tag{20a}$$

for equation (6); and

$$A_j = A_1 + 4(j-1)^2\alpha^2 \qquad j = 2, \ldots, N \tag{20b}$$

for equation (13).

Thus, if $\mathbf{a}^{-1}$ exists, and if A in equations (6) or (13) are given by equations (20a) or (20b), then (14) and (15) are solutions of (6) and (13), respectively, with C_j given by

$$\vec{C} = \mathbf{a}^{-1}\vec{d}. \tag{21}$$

For equation (6), there is a further restriction that C_j give by equation (21) must be positive.

Equations (14)–(21) complete the description of our solutions for equations (6) and (13), for the special case $k^2 = 1$ and the coupled waves being complementary. With the use of transformation (5) and (8), we obtain our N complementary aperiodic solitary-wave solutions for N CNLS and CQ equations (3) and (12). It will be noted that the N aperiodic complementary solitary waves for CNLS equations consist of symmetric (about $\xi = 0$) as well as antisymmetric waves, while those for CQ equations consist of only symmetric waves.

There are other aperiodic-wave solutions. For N CNLS equations, two or more waves may be of the same waveform and they may be waves of order $1, 2, \ldots, N-1$ and an additional solution which consists of N complementary waves of order N. For N CQ equations, two or more waves may be of the same waveform and they may be a subset of waves of order $2, 4, \ldots, 2(N-1)$ and an additional solution which consists of N complementary waves which are a subset of waves of order $2N$. These solutions can be obtained by using a formulation similar to the one leading to equations (9)–(11).

We shall illustrate our results given by equations (14)–(21) first with the examples of $N = 2$ and 3 for equation (6). Using equation (18), we find that equation (6) is analytically solvable for $N = 2$ if b are given by

$$b_{11} = A_1 C_1^{-1} \qquad b_{12} = (A_1 + 2\alpha^2) C_2^{-1} \qquad b_{21} = (A_2 - \alpha^2) C_1^{-1}$$
$$b_{22} = (A_2 + \alpha^2) C_2^{-1} \tag{22a}$$

and the solution is

$$x_1 = \sqrt{C_1}\tanh \alpha t \qquad x_2 = \sqrt{C_2}\,\mathrm{sech}\,\alpha t. \tag{22b}$$

The analytically solvable region (22a) is a small part of the analytically solvable regions given in section 3 and can be obtained by setting $k^2 = 1$ in (IV) and (V). For $N = 3$, equation (6) is analytically solvable if b are given by

$$b_{11} = \tfrac{9}{4} A_1 C_1^{-1} \qquad b_{12} = 3(A_1 + 2\alpha^2) C_2^{-1} \qquad b_{13} = \tfrac{3}{4}(A_1 + 8\alpha^2) C_2^{-1}$$
$$b_{21} = \tfrac{9}{4}(A_2 - \alpha^2) C_1^{-1} \qquad b_{22} = 3(A_2 + \alpha^2) C_2^{-1} \qquad b_{23} = \tfrac{3}{4}(A_2 + 7\alpha^2) C_2^{-1}$$
$$b_{31} = \tfrac{9}{4}(A_3 - 4\alpha^2) C_1^{-1} \qquad b_{32} = 3(A_3 - 2\alpha^2) C_2^{-1} \qquad b_{33} = \tfrac{3}{4}(A_3 + 4\alpha^2) C_2^{-1} \tag{23a}$$

and the solution is

$$x_1 = \sqrt{C_1}(\mathrm{sech}^2\,\alpha t - \tfrac{2}{3}) \qquad x_2 = \sqrt{C_2}\tanh \alpha t\,\mathrm{sech}\,\alpha t \qquad x_3 = \sqrt{C_3}\,\mathrm{sech}^2\,\alpha t. \tag{23b}$$

For the case $b_{ij} = \varepsilon = +1$ or -1, for all $i, j = 1, \ldots, N$, equation (22) for equation (6), $N = 2$, gives

$$C_1 = \varepsilon A_1 \qquad C_2 = \varepsilon(2A_2 - A_1) \qquad \alpha^2 = A_2 - A_1 \qquad A_2 > A_1 > 0\ (\varepsilon = +1)$$
$$A_2 > A_1 < 0\ (\varepsilon = -1) \tag{24}$$

and for $N = 3$, equation (23) gives

$$C_1 = 9\varepsilon A_1/4 \qquad C_2 = 3\varepsilon(2A_2 - A_1) \qquad C_3 = 3\varepsilon(8A_2 - 7A_1)/4 \qquad \alpha^2 = A_2 - A_1$$
$$A_3 = 4A_2 - 3A_1 \qquad A_2 > A_1 > 0\ (\varepsilon = +1) \qquad A_2 > A_1 \geqslant 8A_2/7 < 0\ (\varepsilon = -1). \tag{25}$$

Next, we illustrate our results given by equations (14)–(21) with the examples of $N = 2$ and 3 for the CQ equations, equation (12). Equations (13) are analytically solvable for $N = 2$ with solutions given by equation (15) if b are given by

$$b_{11} = -\tfrac{3}{2} A_1 C_1^{-1} \qquad b_{12} = \tfrac{3}{2}(A_1 + 4\alpha^2) C_2^{-1} \qquad b_{21} = -\tfrac{3}{2}(A_2 - 4\alpha^2) C_1^{-1}$$
$$b_{22} = \tfrac{3}{2} A_2 C_2^{-1} \tag{26a}$$

and the solution is given by

$$x_1 = C_1(\operatorname{sech}^2 \alpha t - \tfrac{2}{3}) \qquad x_2 = C_2 \operatorname{sech}^2 \alpha t. \tag{26b}$$

Equation (13) is analytically solvable for $N = 3$ with solutions given by equation (15) if b are given by

$$b_{11} = \tfrac{35}{8} A_1 C_1^{-1} \qquad b_{12} = -\tfrac{35}{6}(A_1 + 4\alpha^2) C_2^{-1} \qquad b_{13} = \tfrac{35}{24}(A_1 + 16\alpha^2) C_2^{-1}$$
$$b_{21} = \tfrac{35}{8}(A_2 - 4\alpha^2) C_1^{-1} \qquad b_{22} = -\tfrac{35}{6} A_2 C_2^{-1} \qquad b_{23} = \tfrac{35}{24}(A_2 + 12\alpha^2) C_2^{-1}$$
$$b_{31} = \tfrac{35}{8}(A_3 - 16\alpha^2) C_1^{-1} \qquad b_{32} = -\tfrac{35}{6}(A_3 - 12\alpha^2) C_2^{-1} \qquad b_{33} = \tfrac{35}{24} A_3 C_2^{-1} \tag{27a}$$

and the solution is given by

$$x_1 = C_1(\operatorname{sech}^4 \alpha t - \tfrac{8}{7} \operatorname{sech}^2 \alpha t + \tfrac{8}{35}) \qquad x_2 = C_2 \operatorname{sech}^2 \alpha t (\operatorname{sech}^2 \alpha t - \tfrac{6}{7})$$
$$x_3 = C_3 \operatorname{sech}^4 \alpha t. \tag{27b}$$

For the case $b_{ij} = \varepsilon$ for all $i, j = 1, \ldots, N$, equation (26) gives for equation (13), $N = 2$,

$$C_1 = -3\varepsilon A_1/2 \qquad C_2 = 3\varepsilon A_2/2 \qquad \alpha^2 = (A_2 - A_1)/4 \tag{28}$$

and equation (27) gives for equation (13), $N = 3$

$$C_1 = 35\varepsilon A_1/8 \qquad C_2 = -35\varepsilon A_2/6 \qquad C_3 = 35\varepsilon A_3/24 \qquad \alpha^2 = (A_2 - A_1)/4$$
$$A_3 = 4A_2 - 3A_1 \qquad A_2 > A_1 > 0 \text{ (for } \varepsilon = +1) \qquad A_2 > A_1 < 0 \text{ (for } \varepsilon = -1). \tag{29}$$

5. Summary

Starting from the ansatz that analytic solutions for N waves of N CNLS equations can be expressed in terms of N of the $(2n + 1)$ Lamé functions of order n, $n = 1, \ldots, N$, of variable amplitudes and modulus, we have found many regions where the nonlinear coupling parameters can assume wide ranges of values for which these solutions are valid and for which the N coupled equations are thus analytically solvable. If we restrict ourselves to only aperiodic waves, the analytically solvable regions are fewer and smaller. Applications to CQ equations indicate that there may be other coupled nonlinear equations that can be studied in a similar way. We also introduce the concept of classifying sets of solitary waves by their 'order'. For the important case of two CNLS ($N = 2$) equations, for example, the two coupled waves can be chosen from three waves of order one (which reduce to two aperiodic waves, the familiar dark and bright solitary waves) and five waves of order two (which reduce to three aperiodic waves, referred to as red, white and blue solitary waves in [18]). A wave of order N cannot, in general, be a solution of CNLS equations involving $N - 2$ or less coupled fields (or field components). Thus, in particular, any wave of order >2 is not by itself a solitary wave of an NLS equation, i.e. any wave of order >2 must be coupled or accompanied by at least one of its partners of the same order to be solitary waves in coupled NLS equations. This leads to an interesting and potentially useful idea that increasing the number of interacting fields may sometimes facilitate the coupled solitary-wave propagation because of an increased number of choices for the wave-type and amplitudes. The many advantages of using two waves instead of one in a different context were known in other physical problems [19–21]. Besides applications in nonlinear optics, our results may find applications in the study of the dynamics of Bose–Einstein condensates [22] which have attracted considerable interest recently. It is also interesting to note a recent experimental observation of multihump solitons in a dispersive nonlinear medium [23] and the appearance of two of the three waveforms of order two in the theory of incoherent dark solitons [24].

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